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THE ASYMPTOTIC FORM OF THE FUNDAMENTAL SOLUTION OF THE EQUATION OF THE PROPAGATION OF PERTURBATIONS IN A ONE-DIMENSIONAL MEDIUM WITH LOW VISCOSITY†

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Asymptotic expansions of the fundamental solution of the singularly perturbed wave equation, which describes the propagation of perturbations in a viscous medium, are constructed. In particular, a uniform expansion in the region of the wave front, which describes the smoothing of the discontinuity due to the presence of viscosity, is obtained.

1. FORMULATION OF THE PROBLEM

We will assume that the propagation of perturbations in a viscous medium is described by the following equation

$$P\zeta=\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\varepsilon^2\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\right)\zeta=f(x,t)$$

This holds, in particular, for a viscous gas.

When $\varepsilon = 0$ the fundamental solution of the operator P suffers a discontinuity on the wave front: $\zeta = \theta(t-|x|)/2$, where θ is Heaviside's function. When $\varepsilon \neq 0$ it is continuous, and hence as $\varepsilon \to 0$ it should be a function of the type of boundary layer in the neighbourhood of the wave front. Below we construct asymptotic expansions of the fundamental solution of the operator B as $t \to +0$ and as $\omega = t/\varepsilon^2 \to \infty$, $\lambda = x/t = \text{const}$. In the first case, we use the theorem of the expansion of a transform, and in the second we use the method of steepest descent.

2. THE ASYMPTOTIC FORM AS $t \to +0$

Suppose ζ is the fundamental solution of the slow increase in the operator P

$$P\zeta = \delta(x,t) \tag{2.1}$$

We apply a Fourier transformation with respect to x to Eq. (2.1), we obtain a solution of the corresponding ordinary differential equation [1, p. 200] and we use the formula for inverting a Fourier transform. We obtain

$$\zeta=\frac{\theta(t)}{\pi}\int_0^{+\infty}\exp\left(-\frac{\varepsilon^2}{2}r^2t\right)\frac{\operatorname{sh}\alpha_2(r)t}{\alpha_0(r)}\cos xr\,dr \tag{2.2}$$

$$\alpha_0(r)=\left(\frac{\varepsilon^4}{4}r^4-r^2\right)^{1/2}$$

It can be shown, by integration by parts, that $\zeta = O(x^{-N})$ for any N, $x \to \pm\infty$. Henceforth we will assume that $x > 0$.

We apply a Laplace transformation to relation (2.2)

$$\int_0^{+\infty}\exp(-pt)\zeta dt=\frac{1}{\pi}\int_0^{+\infty}\frac{\cos xr}{p^2+\left(1+\varepsilon^2p\right)r^2}dr=\frac{\exp\left(-xp\left(1+\varepsilon^2p\right)^{-1/2}\right)}{2p\left(1+\varepsilon^2p\right)^{1/2}}$$

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The validity of the change in the order of integration follows from the absolute convergence of the repeated integral and Fubini's theorem. We further use the formulae from [2, p. 264] and (3.723.2) from [3, p. 420].

Using the inversion formula taking into account the replacement $p = (p-1)/\varepsilon^2$ we obtain

$$\zeta = \frac{1}{4\pi i}\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \exp\left(\frac{(p-1)t}{\varepsilon^2}\right)\exp\left(-\frac{xp^{1/2}}{\varepsilon^2}\right)\frac{\exp\left(\varepsilon^{-2}xp^{-1/2}\right)}{(p-1)p^{1/2}}dp \tag{2.3}$$

Substituting the asymptotic expansion of the last factor in the integrand at an infinitely distant point into (2.3) and integrating term by term, in accordance with Eq. (9.397) of [4, p. 433], we obtain

$$\zeta \sim \exp\left(-\frac{1}{\varepsilon^2}\left(t + \frac{x^2}{8t}\right)\right)\sum_{k=0}^{\infty}\frac{2^{1/2}C_k t^{1/2+1/2}}{\sqrt{\pi}\varepsilon^{k+1}}D_{-k-2}\left(\frac{x}{\varepsilon\sqrt{2t}}\right), \quad t \to +0 \tag{2.4}$$

$$C_k = \sum_{\substack{m+2n=k \\ m,n\geq 0}} \frac{x^m}{m!\varepsilon^{2m}}$$

where D_{-k} are parabolic cylinder functions.

Relation (9.246.1) of [3, p. 1079] shows that the terms of the series (2.4) form an asymptotic scale. An estimate of the residue, from which it follows that expansion (2.4) is in fact asymptotic, was made in [5, p. 326].

One term of this expansion gives

$$\zeta = \frac{2\varepsilon}{\sqrt{\pi}}\frac{t^{3/2}}{x^2}(1 + O(t))\exp\left(-\frac{x^2}{4\varepsilon^2 t} - \frac{t}{\varepsilon^2}\right)$$

3. THE SADDLE POINT AND THE LINE OF STEEPEST DESCENT

Making the change $z = \sqrt{p}$ in (2.3), where we have taken the analytical extension of the main term of the root from the real axis in the right half-plane, we obtain

$$\zeta = \frac{1}{2\pi i}\int_C \frac{E(z,\omega)}{z^2 - 1}dz, \quad E(z,\omega) = \exp(\omega q(z)) \tag{3.1}$$

where

$$\lambda = x/t > 0, \quad \omega = t/\varepsilon^2, \quad q(z) = z^2 - 1 - \lambda(z - 1/z)$$

and the contour C is the image of the vertical straight line for this change of variables.

The saddle points can be found from the equation

$$z^3 - \lambda z^2/2 - \lambda/2 = 0 \tag{3.2}$$

Equation (3.2) has a unique real root $z_1 > 0$. Below we will assume that $q(z_1) < 0$, when $\lambda \neq 1$, $q''(z_1) > 0$, and also that the function $z_1 = z_1(\lambda)$ is monotonically increasing and $z_1(1) = 1$.

We obtain the following asymptotic representation for z_1 using Newton's diagram or directly from Cardano's formula

$$z_1 = \left(\frac{\lambda}{2}\right)^{1/3} + \frac{\lambda}{6} + \ldots, \quad \lambda \to 0;$$

$$z_1 = \frac{\lambda}{2} + \frac{2}{\lambda} + \ldots, \quad \lambda \to \infty; \quad z_1 = 1 + \frac{\lambda - 1}{2} + \ldots \quad \lambda \to 1$$

Suppose $z = \xi + i\eta$. The equation of the line of steepest descent L can be found from the relations $\operatorname{Im} q(z) = \operatorname{Im} q(z_1)$, $\operatorname{Re} q(z) < \operatorname{Re} q(z_1)$

$$F(\xi,\eta) = (2\xi - \lambda)(\xi^2 + \eta^2) - \lambda = 0 \tag{3.3}$$

It follows from Eq. (3.3) that $\xi > \lambda/2$ and that the line L is symmetrical about the real axis. In addition, L allows of the explicit representation $\xi = \xi(\eta)$ since $\partial F/\partial\xi > 0$. The function $\xi = \xi(\eta)$ is monotonically increasing when $\eta < 0$ and monotonically decreasing when $\eta > 0$.

When $z_1 > 1$ ($\lambda > 1$) the contour C can be deformed into L. In fact, on the arcs of circles of radius R situated between C and L (see Fig. 1), we have

$$\operatorname{Re} q(z) = R^2 \cos 2\varphi - \lambda R \cos\varphi + O(1) < 0$$

for sufficiently large $R(R = |z|, \varphi = \arg z)$. When $z_1 < 1$ ($\lambda > 1$), when C is deformed into L we must once again take into account the residue of the integrand with respect to the point $z = 1$

$$\operatorname{Res}\left(\frac{E(z,\omega)}{z^2-1},\ 1\right) = \frac{1}{2} \tag{3.4}$$

4. THE ASYMPTOTIC FORM AS $\omega \to \infty$

Consider the equation

$$q(z) - q(z_1) = -u^2 \tag{4.1}$$

By the inverse-function theorem a holomorphic function $z = z(u)$ exists, defined in a certain region of the origin of coordinates in the u plane, which turns (4.1) into an identity and such that

$$z(0) = z_1,\quad z'(0) = i(2/q''(z_1))^{1/2}$$

Here and below the root of a real positive number is understood in the sense of the principal value.

Suppose δ is a small positive number. We deform the contour C into L and split it into two parts: $L_\delta \ni z_1$ is the part of the contour of length δ and $L \backslash L_\delta$. To evaluate the integral over L_δ we make the replacement $z = z(u)$ in (3.1). Estimating the integral with respect to $L \backslash L_\delta$ we obtain

$$\zeta = \frac{E(z_1,\omega)}{2\pi i}\int_{-\alpha}^{\beta} \exp(-\omega u^2) G(u)\,du + \frac{1}{2}\theta(1-\lambda) + O(E(z_1,\omega), \exp(-\omega\gamma)),\quad G(u) = \frac{1}{(z^2-1)}\frac{dz}{du} \tag{4.2}$$

where α, β, γ are certain positive numbers.

The second term on the right-hand side of (4.2) is supplemented in accordance with formula (3.4).

Using Watson's lemma [6, p. 57] we obtain

$$\zeta = \frac{1}{2}\theta(t-x) + \frac{E(z_1,\omega)}{\omega^{1/2}}\left(\frac{1}{(z_1^2-1)(2\pi q''(z_1))^{1/2}} + O\left(\frac{1}{\omega}\right)\right) \tag{4.3}$$

The asymptotic form (4.3) is non-uniform in the region of the front, i.e. $\lambda \approx 1$ ($z_1 \approx 1$). To obtain a uniform expansion we represent the function $G(u)$ in the form

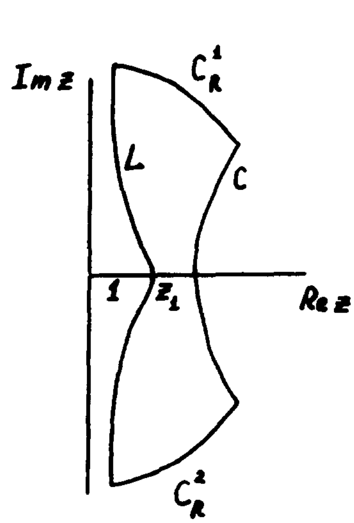

Fig. 1.

$$G(u)=\frac{1}{2(u-b)}+T(u) \tag{4.4}$$

where $b=\sqrt{(-q(z_1))}i\ \mathrm{sign}\,(\lambda-1)$, while the function $T(u)$ is holomorphic in a certain neighbourhood of the origin of coordinates for all $\lambda\approx 1$.

We therefore have [6, p. 525]

$$\int_{-\infty}^{+\infty}\frac{\exp\left(-\omega u^2\right)}{u-b}du=i\pi E(z_1,-\omega)\left(\mathrm{erfc}\left(-ib\omega^{1/2}\right)-2\theta(1-\lambda)\right)$$

Substituting (4.4) into (4.2) and using the last relation we obtain

$$\zeta=\frac{1}{4}\mathrm{erfc}\left(-ib\omega^{1/2}\right)+\frac{E(z_1,\omega)}{2\pi i}\int_{-\alpha}^{\beta}\exp\left(-\omega u^2\right)T(u)du+O\left(E(z_1,\omega)\exp(-\omega\gamma)\right)$$

To calculate the main term of the expansion we gain use Watson's lemma

$$\zeta=\frac{1}{4}\mathrm{erfc}\left(-ib\omega^{1/2}\right)+\frac{E(z_1,\omega)}{\omega^{1/2}}\left(\frac{1}{\left(z_1^2-1\right)\left(2\pi q''(z_1)\right)^{1/2}}+\frac{1}{4\sqrt{\pi}ib}+O\left(\frac{1}{\omega}\right)\right) \tag{4.5}$$

Note that it follows from the asymptotic form of the function erfc (x) as $x\to\pm\infty$ that for fixed $\lambda\neq 1$ formula (4.5) can be replaced by (4.3).

In the limit as $\lambda\to 1$ we obtain in (4.5) the power asymptotic form on the front

$$\zeta=\frac{1}{4}+\frac{1}{16\sqrt{2\pi}}\frac{1}{\omega^{1/2}}+O\left(\frac{1}{\omega^{3/2}}\right),\quad \lambda=1$$

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